Nonequilibrium phase transition in the coevolution of networks and opinions

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(Received 9 March 2006; revised manuscript received 27 September 2006; published 10 November 2006)

Models of the convergence of opinion in social systems have been the subject of considerable recent attention in the physics literature. These models divide into two classes, those in which individuals form their beliefs based on the opinions of their neighbors in a social network of personal acquaintances, and those in which, conversely, network connections form between individuals of similar beliefs. While both of these processes can give rise to realistic levels of agreement between acquaintances, practical experience suggests that opinion formation in the real world is not a result of one process or the other, but a combination of the two. Here we present a simple model of this combination, with a single parameter controlling the balance of the two processes. We find that the model undergoes a continuous phase transition as this parameter is varied, from a regime in which opinions are arbitrarily diverse to one in which most individuals hold the same opinion.

DOI: 10.1103/PhysRevE.74.056108 PACS number(s): 89.75.Fb, 87.23.Ge, 64.60.Ak, 89.75.Hc

I. INTRODUCTION

Simple mathematical models describing emergent phenomena in human populations have a long history of study in the social sciences [1]. It is only relatively recently, however, that physicists have noted the close conceptual and mathematical connections between these models and traditional models in statistical physics such as spin models. Building on this observation, there have been a number of important advances in the understanding of social models in the last few years, most notably for models of social networks [2]. Much of the work in this area has been directed at improving our understanding of network structure but there has also been a substantial line of investigation focusing on dynamical processes on networks. One example, which has a long history in economics and sociology but which is also well suited to study using physics methods, is the dynamics of opinion formation. This problem highlights one of the fundamental questions in network dynamics, namely, whether the dynamics taking place on a network controls the network structure or the structure controls the dynamics.

It is observed that real social networks tend to divide into groups or communities of like-minded individuals [3]. An obvious question to ask is whether individuals become like-minded because they are connected via the network [4–10], or whether they form network connections because they are like-minded [3]. Both situations have been modeled using physics-style methods, the first with opinion formation models [4–7] and the second with models of "assortative mixing" or "homophily" [11]. In the real world, of course, both mechanisms may be in effect at once—the network changing in response to opinion and opinion changing in response to the network [12]. In this paper we study a simple model—perhaps the simplest—that combines opinion dynamics with assortative network formation, revealing an apparent phase transition between regimes in which one process or the other dominates the dynamics.

Our work is based on the voter model of opinion formation, which was independently proposed both as a model of biological population dynamics [4] and as an iterative version of an economic model of "public choice" [5]. The model has substantial experimental support in both areas [13,14]. Consider a network of N vertices, representing individuals, joined in pairs by M edges, representing active acquaintances between individuals [15]. The number of edges M is fixed, reflecting the fact that individuals can only maintain a limited number of connections at a given moment. This also implies that the network is sparse: the average number of connections an individual has is constant as N becomes large.

Each individual is assumed to hold one of G possible opinions on some topic of interest. The opinion of individual i is denoted g_i. In the past, researchers have considered both cases where G is a fixed small number, such as a choice between candidates in an election [6–8], and cases in which the number of possible opinions is essentially unlimited [9,16], so that G can be arbitrarily large. An example of the latter might be religious belief (or lack of it)—the number of subtly different religious beliefs appears to be limited only by the number of people available to hold them.

The case of fixed small G has relatively simple behavior compared to the case of arbitrarily large G, and so it is on the latter that we focus here. We will assume that the number of possible opinions scales in proportion to the number of individuals, and parametrize this proportionality by the ratio $\gamma=N/G$. (It is possible that not all opinions will end up existing in the population. Our model allows for some opinions to become extinct as the dynamics evolves, so that the final number of distinct opinions may be less than G.)

II. DEFINITION OF THE MODEL

The M edges of the network are initially placed uniformly at random between vertex pairs, and opinions are assigned to vertices uniformly at random. We then study by computer simulation a dynamics in which on each step of the simulation we either move an edge to lie between two individuals whose opinions agree, or we change the opinion of an individual to agree with the opinion of one of their neighbors. To be specific, on each step we pick a vertex i at random. If the

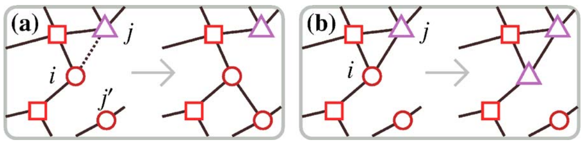

FIG. 1. (Color online) An illustration of the model, with vertex shapes representing different opinions. At each time step the system is updated according to the process illustrated in panel (a) with probability ϕ or panel (b) with probability $1-\phi$. In (a) a vertex i is selected at random and one of its edges—in this case the edge (i,j)—is rewired to a new vertex j' holding the same opinion as i. In (b) vertex i adopts the opinion of one of its neighbors j.

degree k_i of that vertex is zero, we do nothing. Otherwise, we do the following (see Fig. 1).

(1) With probability ϕ, we select at random one of the edges attached to i and move the other end of that edge to a vertex chosen randomly from the set of all vertices having opinion g_i.

(2) Otherwise (i.e., with probability $1-\phi$) we pick a random neighbor j of i and set g_i equal to g_j.

Step 1 represents the formation of new acquaintances between people of similar opinions. Step 2 represents the influence of acquaintances on one another, opinions becoming similar as a result of acquaintance.

Note that both the total number of edges M in our network and the total number of possible opinions G are fixed. In the limit of large system size, the model thus has three parameters: the average degree $\bar{k}=2M/N$, the mean number of people holding a particular opinion $\gamma=N/G$, and the parameter ϕ. In our studies, we typically keep the first two of these parameters fixed and ask what happens as we vary the third.

III. NUMERICAL RESULTS

The expected qualitative behavior of the model is clear. Since both of our update moves tend to decrease the number of nearest-neighbor vertex pairs with different opinions, we should ultimately reach a state in which the network is divided into a set of separate components, disconnected from one another, with all members of a component holding the same opinion. That is, the model segregates into a set of communities such that no individual has any acquaintances with whom they disagree. We call this the "consensus state." Furthermore, once we reach the consensus state, all moves in the model involve the random rearrangement of edges within components, and hence, in the limit of long time, the components become random graphs with uniform uncorrelated arrangements of their edges.

The primary interest in our model is therefore in the number and size of the communities that form and in the dynamics of the model as it comes to consensus. Let us consider the distribution $P(s)$ of the sizes s of the consensus communities. In the limit $\phi\to 1$, only updates that move edges are allowed and hence the consensus state is one in which the communities consist of the sets of initial holders of the individual opinions. Since the initial assignment of opinions is random,

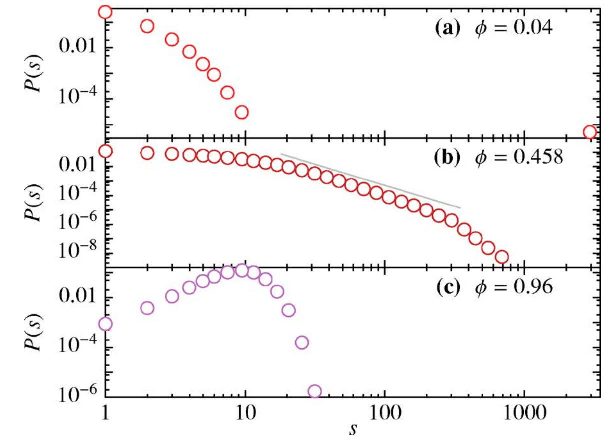

FIG. 2. (Color online) Histograms of community sizes in the consensus state for values of ϕ above, at, and below the critical point in panels (a), (b), and (c), respectively. Values of the other parameters are $N=3200$, $M=6400$ (giving $\bar{k}=4$), and $\gamma=10$. In panel (b) the distribution appears to follow a power law for part of its range, with exponent 3.5 ± 0.3, as indicated by the solid line. Numerical data are averaged over 10^4 realizations for each value of ϕ and binned logarithmically.

the sizes of these sets follow the multinomial distribution, or the Poisson distribution with mean γ in the limit of large N. Conversely, in the limit $\phi\to 0$, only changes of opinion are allowed and not edge moves, which means that the communities correspond to the initial components in the graph, which are simply the components of a random graph. Assuming we are in the regime $\bar{k}>1$ in which a giant component exists in the random graph, we will then have one giant (extensive) community and an exponential distribution of small communities. Thus, in varying ϕ we go from a situation in which we have only small communities with constant average size γ to one in which we have a giant community plus a set of small ones. (Naturally, if the we are in the regime $\bar{k}<1$ in which no initial giant component exists, then we will not see this behavior. Real acquaintance networks, however, typically have average degree far above 1 and at least one large component, so it is the $\bar{k}>1$ case studied here that is of primary interest.)

This is the classic behavior seen in a system undergoing a continuous phase transition and it leads us to conjecture that our model displays a phase transition with decreasing ϕ at which a giant community of like-minded individuals forms. In other words, there is a transition from a regime in which the population holds a broad variety of views to one in which most people believe the same thing. We now offer a variety of further evidence to support this conjecture. (Phase transition behavior is also seen in some models of opinion formation on static networks, such as the model of Ref. [10], although the mechanisms at work appear to be different from those considered here.)

In Fig. 2 we show plots of $P(s)$ from simulations of our model for $\bar{k}=4$ and $\gamma=10$. As the figure shows, we do indeed see a qualitative change from a regime with no giant community to one with a giant community. At an intermediate value of ϕ around 0.46 we find a distribution of community sizes that appears to follow a power law $P(s) \sim s^{-\alpha}$ over a significant part of its range, another typical signature of criticality. (The particular value $\phi = 0.458$ is taken from the finite size scaling analysis below. Plots for other values close to 0.46 are rather similar.) The exponent α of the power law is measured to be 3.5 ± 0.3, incompatible with the value 2.5 of the corresponding exponent for the distribution of components in a random graph at the phase transition marking the formation of a giant component (a transition that belongs to the mean-field percolation universality class).

Further light can be shed on the transition in our model by performing a finite size scaling analysis. To do this, we need first to choose an order parameter for the model. The obvious choice is the size S of the largest community in the consensus state as a fraction of system size. The arguments above suggest that this quantity should be of size $O(N^{-1})$ [or possibly $O(N^{-1} \ln N)$] for values of ϕ above the phase transition (and hence zero in the thermodynamic limit) and O(1) below it. We assume a scaling relation of the form

$$S = N^{-a} F[N^b(\phi - \phi_c)], \qquad (1)$$

where ϕ_c is the critical value of ϕ (which is presumably a function of $\bar{k}$ and γ), F is a universal scaling function (bounded as its argument tends to $\pm\infty$), and a and b are critical exponents. To estimate ϕ_c we plot $N^a S$ against ϕ and tune a such that the results for simulations at different N but fixed $\bar{k}$ and γ cross at a single point, which is the critical point. Such a plot for $\bar{k}=4$ and $\gamma=10$ is shown in Fig. 3(a). With $a = 0.61 \pm 0.05$ we obtain a unique crossing point at $\phi_c = 0.458 \pm 0.008$, which agrees well with rough estimates of ϕ_c from histograms such as Fig. 2.

Using this value we can now determine the exponent b by plotting $N^a S$ against $N^b(\phi - \phi_c)$. Since $F(x)$ is a universal function, we should, for the correct choice of b, find a data collapse in the critical region. In Fig. 3(b) we show that such a data collapse does indeed occur for $b = 0.7 \pm 0.1$. We have performed similar finite size scaling analyses for a variety of other points $(\bar{k}, \gamma)$ in the parameter space and, as shown in Fig. 4, we find that the position ϕ_c of the phase transition varies but that good scaling collapses exist in all cases for values of the critical exponents consistent with the values $a = 0.61$ and $b = 0.7$ found above.

Despite the qualitative similarities between the present phase transition and the random graph (percolation) transition, our exponent values for a and b show that the two transitions are in different universality classes: the corresponding exponents for the random graph are $a = b = \frac{1}{3}$, which are incompatible with the values measured above.

Our model differs from percolation in another important respect also: percolation is a static, geometric phase transition, whereas the present model is fundamentally dynamic, the consensus arising as the limiting fixed point of a converging nonequilibrium dynamics. It is interesting therefore to explore the way in which our model approaches consensus.

In previous studies of opinion formation models of this type on fixed networks, a key quantity of interest is the average convergence time τ, which is the number of updates per vertex needed to reach consensus. If $\phi = 0$ then τ is

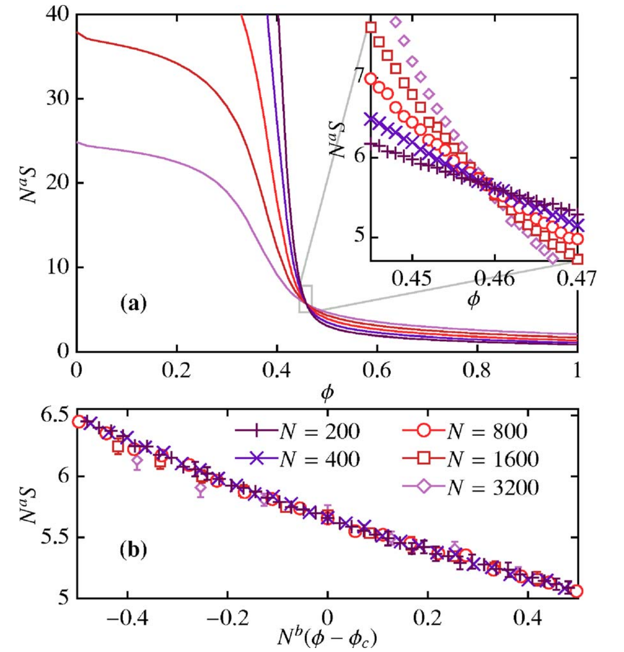

FIG. 3. (Color online) Finite size scaling for $\bar{k}=4$ and $\gamma=10$. (a) Crossing plot used to determine the critical value ϕ_c and exponent a. We find $\phi_c = 0.458 \pm 0.008$ and $a = 0.61 \pm 0.05$. The inset shows a blow up of the region around the critical point. (b) Scaling collapse used to determine the exponent b, which is found to take the value $b = 0.7 \pm 0.1$. The data are averaged over 10^4 realizations for each value of ϕ. Error bars are shown where they are larger than the symbol size.

known to scale as $\tau \sim N$ as system size becomes large [7]. In the opposite limit ($\phi = 1$), opinions are fixed and convergence to consensus involves moving edges one by one to fall between like-minded pairs of individuals. This is a standard sampling-with-replacement process in which the number U of unsatisfied edges is expected to decay as $U \sim M e^{-t/M}$ for large times t. Thus the time to reach a configuration in which $U = O(1)$ is $t \sim M \ln M$, and the convergence time is this quantity divided by the system size N. For fixed average degree $\bar{k} = 2M/N$, this then implies that $\tau \sim \ln N$. This result is confirmed numerically in Fig. 5(a).

For ϕ close to ϕ_c, experience with other phase transitions leads us to expect critical fluctuations and critical slowing down in τ. Figure 5(b) shows that indeed there are large

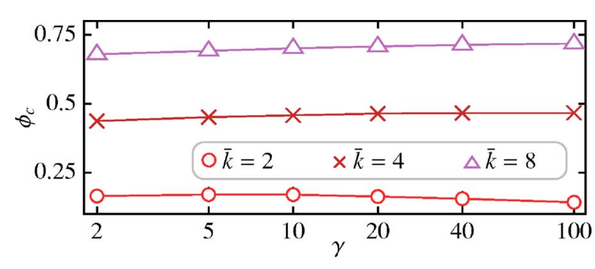

FIG. 4. (Color online) Values of ϕ_c as a function of γ for various $\bar{k}$ obtained by finite size scaling analyses using system sizes $N = 200$, 400, 800, and 1600, and 10^4 realizations for each size and set of parameter values. Note that the horizontal axis is logarithmic.

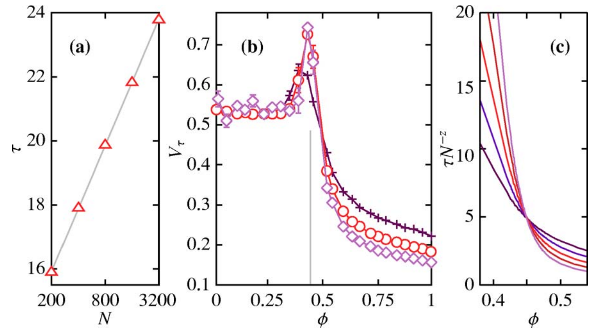

FIG. 5. (Color online) Scaling of the average time τ to reach consensus. (a) Convergence time as a function of system size for $\phi=1$. The straight line is a fit to a logarithmic form and indicates that $\tau \sim \ln N$. (b) Coefficient of variation of the convergence time as a function of ϕ. The vertical gray line denotes our best estimate of the critical point $\phi_c=0.458$. (c) Scaling plot used to determine the dynamical exponent z. The crossing point falls at $\phi=0.45\pm0.02$, consistent with our other estimates of ϕ_c. The dynamical exponent is found to take the value $z=0.61\pm0.15$. Parameter values are $\bar{k}=4$ and $\gamma=10$ in all panels. All data points are averaged over 10^4 realizations. Symbols are the same as in Fig. 3. For the sake of clarity, system sizes $N=400$ and $N=1600$ are omitted in (b).

fluctuations in the convergence time in the critical region. The figure shows the value of the coefficient of variation V_τ of the consensus time (i.e., the ratio of the standard deviation of τ to its mean) as a function of ϕ and a clear peak is visible around $\phi_c \simeq 0.46$. To characterize the critical slowing down we assume that τ takes the traditional scaling form $\tau \sim N^z$ at the critical point, where z is a dynamical exponent [17]. Figure 5(c) shows a plot of τN^{-z} as a function of ϕ. If the system follows the expected scaling at ϕ_c then the resulting curves should cross at the critical point. Although good numerical results are considerably harder to obtain in this case than for the community sizes presented earlier, we find that the curves cross at a single point if $z=0.61\pm0.15$ and $\phi=0.44\pm0.03$, the latter being consistent with our previous value of $\phi_c=0.46$ for the position of the phase transition.

IV. CONCLUSIONS

We have proposed a simple model for the simultaneous formation of opinions and social networks in a situation in which both adapt to the other. Our model contrasts with earlier models of opinion formation in which social structure is regarded as static and opinions are an outcome of that preexisting structure [10,18]. Our model is a dynamic, nonequilibrium model that reaches a consensus state in finite time on a finite network. The structure of the consensus state displays clear signatures of a continuous phase transition as the balance between the two processes of opinion change and network rewiring is varied. We have demonstrated a finite size scaling data collapse in the critical region around this phase transition, characterized by universal critical exponents independent of model parameters. The approach to the consensus state displays critical fluctuations in the time to reach consensus and critical slowing down associated with an additional dynamical exponent. The phase transition in the model is of particular interest in that it provides an example of a simple process in which a fundamental change in social structure can be produced by only a small change in the parameters of the system.

Finally, we note that for the specific example of opinion formation mentioned in the Introduction—that of choice between religions—it is known that the sizes of the communities of adherents of religious beliefs are in fact distributed, roughly speaking, according to a power law [19]. This may be a signature of critical behavior in opinion formation, as displayed by the model described here, although other explanations, such as the Yule process [20], are also possible.

ACKNOWLEDGMENTS

The authors thank Paul Krapivsky and Claudio Castellano for helpful suggestions and comments. This work was supported in part by the Wenner-Gren Foundations (PH) and the National Science Foundation under Grant No. DMS-0405348 (MEJN).

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with multiedges and self-edges disallowed, we have in the interest of simplicity allowed multiedges and self-edges in our calculation. Since these form only a small fraction of all edges, we expect that our results would change little if we were to remove them.

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